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Varieties generated by modes of submodes

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Dedicated to Professor Anna Romanowska

Abstract. In a natural way, we can "lift" any operation defined on a set A to an operation on the set of all non-empty subsets of A and obtain from any algebra (A, Ω) its *power algebra* of subsets. G. Grätzer and H. Lakser proved that for a variety $\mathcal{V}$, the variety $\mathcal{V}\Sigma$ generated by power algebras of algebras in $\mathcal{V}$ satisfies precisely the consequences of the linear identities true in $\mathcal{V}$. For certain types of algebras, the sets of their subalgebras form subalgebras of their power algebras. They are called *the algebras of subalgebras*. In this paper, we partially solve a long-standing problem concerning identities satisfied by the variety $\mathcal{VS}$ generated by algebras of subalgebras of algebras in a given variety $\mathcal{V}$. We prove that if a variety $\mathcal{V}$ is idempotent and entropic and the variety $\mathcal{V}\Sigma$ is locally finite, then the variety $\mathcal{VS}$ is defined by the idempotent and linear identities true in $\mathcal{V}$.

1. Introduction

For a given set A, denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A. For any n-ary operation $\omega\colon A^n \to A$, we define *the complex* (*or power*) *operation* $\omega\colon \mathcal{P}_{>0}A^n \to \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \dots, A_n) := \{\omega(a_1, \dots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \dots, A_n \subseteq A$. *The power* (*complex* or *global*) *algebra* of an algebra (A, Ω) is the algebra $(\mathcal{P}_{>0}A, \Omega)$.

The complex operation is a natural generalization of the multiplication of cosets of a subgroup of a group, which was introduced by Frobenius. Besides group theory, power operations appeared also in other algebraic theories. For example, the set of ideals of a distributive lattice $(L, \vee, \wedge)$ again forms a lattice, where meets and joins are precisely the power operations of $\vee$ and $\wedge$. In formal language theory, the product of two languages is the power operation of concatenation of words.

B. Jónsson and A. Tarski [9, 10] first applied the construction of power structures to investigate Boolean algebras with operators. Later, power algebras

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were studied by several authors, for instance by A. Shafaat [22], V. Trnková [24], A. Szendrei [23], J. Ježek [8], G. Grätzer and S. Whitney [7], G. Grätzer and H. Lakser [6], R. Goldblatt [4], C. Brink [3], I. Bošnjak and R. Madarász [2], and many others.

Closely related to power algebras of sets are complex algebras of subalgebras. Let AS be the set of all (non-empty) subalgebras of (A, Ω). In general, the family AS does not have to be closed under complex operations. However, if it is, (AS, Ω) is a subalgebra of the algebra $(\mathcal{P}_{>0}A, \Omega)$ and is called *the algebra of subalgebras* of (A, Ω). For example, if an algebra (A, Ω) is *entropic*, i.e., any two of its operations commute, then its algebra of subalgebras is always defined.

Some properties of an algebra (A, Ω) may remain invariant under power construction, but obviously not all of them. In particular, not all identities true in (A, Ω) will be satisfied in $(\mathcal{P}_{>0}A, \Omega)$ or in (AS, Ω). For example, the power algebra of a group is not again a group [6].

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{VS}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{VS} := \mathrm{HSP}(\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

This happens, for example, in the case $\mathcal{V}$ is *entropic*, i.e., consists of entropic algebras. It is clear that $\mathcal{VS} \subseteq \mathcal{V}\Sigma$. Moreover, $\mathcal{V} \subseteq \mathcal{V}\Sigma$ because every algebra (A, Ω) can be embedded into $(\mathcal{P}_{>0}A, \Omega)$ by $x \mapsto \{x\}$.

G. Grätzer and H. Lakser [6] determined the identities satisfied by the variety $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

We call a term t of the language of a variety $\mathcal{V}$ *linear* if every variable occurs in t at most once. An identity $t \approx u$ is called *linear* if both terms t and u are linear.

Theorem 1.1 ([6]). *Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}\Sigma$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.*

Corollary 1.2 ([6]). *Let $\mathcal{V}$ be a variety of algebras. Then $\mathcal{V}\Sigma = \mathcal{V}$ if and only if $\mathcal{V}$ is defined by a set of linear identities.*

A similar general characterization for varieties $\mathcal{VS}$ is still not known. Though $\mathcal{VS}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{VS}$.

Recall that an algebra (A, Ω) is *idempotent* if each singleton is a subalgebra, i.e., the following identities are satisfied in (A, Ω) for every n-ary $\omega \in \Omega$:

$$\omega(x, \ldots, x) \approx x \qquad \text{(idempotent law).}$$

A variety $\mathcal{V}$ of algebras is called *idempotent* if every algebra in $\mathcal{V}$ is idempotent.

Note that the property of entropicity may also be expressed by means of identities:

$$\begin{aligned} &\omega(\varphi(x_{11},\dots,x_{n1}),\dots,\varphi(x_{1m},\dots,x_{nm})) \\ \approx\ &\varphi(\omega(x_{11},\dots,x_{1m}),\dots,\omega(x_{n1},\dots,x_{nm})) \qquad \text{(entropic law)}, \end{aligned}$$

for every m-ary $\omega \in \Omega$ and n-ary $\varphi \in \Omega$.

It is known that for an idempotent and entropic variety $\mathcal{V}$, the variety $\mathcal{VS}$ is also idempotent and entropic. (See [17], [19], [20] and [18, Sections 1.4 and 3.1].) Since entropic identities are linear, it follows that in this case, the variety $\mathcal{V}\Sigma$ is entropic too, but very rarely is again idempotent.

By Theorem 1.1, an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$. On the other hand, if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathcal{VS} \subseteq \mathcal{IV} \subseteq \mathcal{V}\Sigma$, where $\mathcal{IV}$ is the idempotent subvariety of $\mathcal{V}\Sigma$. But the inclusion $\mathcal{V} \subseteq \mathcal{VS}$ does not hold in general. For example, for the variety $\mathcal{A}$ of Abelian groups, $(A, \cdot, ^{-1})$, defined as inverse semigroups, $\mathcal{AS}$ is idempotent and entropic [13], whence $\mathcal{A} \nsubseteq \mathcal{AS}$. This example also shows that the variety $\mathcal{VS}$ can be idempotent while $\mathcal{V}$ is not. If a variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{VS} \subseteq \mathcal{V}$. Hence, if an idempotent variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{VS} = \mathcal{V}$.

The following conjecture was stated in [12]; see also [15] and [1].

Conjecture 1.3. *An idempotent variety $\mathcal{V}$, in which every algebra has the algebra of subalgebras, coincides with $\mathcal{VS}$ if and only if $\mathcal{V}$ has a basis consisting of idempotent and linear identities.*

This statement for non-idempotent varieties is false. Let $\mathcal{V}$ be the non-idempotent variety of entropic groupoids satisfying $(xx)y \approx xy$ and $y(xx) \approx yx$. It was shown in [1] that $\mathcal{V} = \mathcal{VS}$ and that these two non-linear identities cannot be deduced from any set of linear identities true in $\mathcal{V}$.

Hence, very natural classes for investigating algebras of subalgebras are varieties of *modes*—idempotent and entropic algebras. Modes and algebras of subalgebras of modes were introduced and investigated in detail by A. Romanowska and J. D. H. Smith [19, 20, 18, 21].

In this paper, we partially solve a long-standing problem in the theory of modes concerning the identities true in the varieties $\mathcal{VS}$ and prove the following theorem.

Main Theorem. *Let $\mathcal{M}$ be a variety of modes and let the variety $\mathcal{M}\Sigma$ be locally finite. The variety $\mathcal{MS}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.*

As an immediate consequence, we obtain the confirmation of Conjecture 1.3 in the case of any variety $\mathcal{M}$ of modes for which $\mathcal{M}\Sigma$ is locally finite. Our proof is based on the observation that any quotient of an algebra (A, Ω) from locally finite variety $\mathcal{M}\Sigma$ that lies in $\mathcal{IM}$ actually lies in the subvariety $\mathcal{MS}$.

The main idea of the proof is to extend the language of power algebras by adding an operation of union of sets. Then, factoring by a specific congruence, we obtain an idempotent factor which is a modal (i.e., a semilattice ordered mode) in the case the initial algebra was a mode (see Theorem 3.3). Finally, we restrict the language to the one of the original mode variety and consider the identities that do not include the operation of union.

Note that earlier attempts to prove Conjecture 1.3, which provided only some partial solutions, were based on a completely different approach. (See e.g. [1].)

The paper is organized as follows. In Section 2, we introduce the concept of extended power algebras—power algebras with one additional union operation. We also investigate the smallest congruence of an extended power algebra of a mode that gives an idempotent factor. Two descriptions of such an idempotent replica congruence are presented. Here the assumption that all congruences are congruences also with respect to the operation $\cup$ is essential (see Theorems 2.2 and 2.5). In Section 3, we prove our main result.

For more information about modes, we refer the reader to the monographs [18] and [21]. The set of all equivalence classes of an equivalence relation $\varrho \subseteq A \times A$ is denoted by A^{ϱ}. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0.

2. The idempotent replica congruence

Let (A, Ω) be an algebra. The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. By adding the operation $\cup$ to the set of fundamental operations of the power algebra of (A, Ω), we obtain *the extended power algebra* $(\mathcal{P}_{>0}A, \Omega, \cup)$.

B. Jónsson and A. Tarski [9] proved that complex operations *distribute* over the union $\cup$, i.e., for each n-ary operation $\omega \in \Omega$ and non-empty subsets $A_1, \ldots, A_i, \ldots, A_n, B_i$ of A,

$$\begin{aligned}\omega(A_1, \ldots, A_i \cup B_i, \ldots, A_n)\\ = \omega(A_1, \ldots, A_i, \ldots, A_n) \cup \omega(A_1, \ldots, B_i, \ldots, A_n),\end{aligned} \tag{2.1}$$

for any $1 \leq i \leq n$.

Power algebras also have the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and $A_{ij} \subseteq A$, for $1 \leq i \leq n$ and $1 \leq j \leq r$:

$$\omega(A_1, \ldots, A_n) \subseteq \omega(B_1, \ldots, B_n), \tag{2.2}$$

$$\begin{aligned}\omega(A_{11}, \ldots, A_{n1}) \cup \cdots \cup \omega(A_{1r}, \ldots, A_{nr})\\ \subseteq \omega(A_{11} \cup \cdots \cup A_{1r}, \ldots, A_{n1} \cup \cdots \cup A_{nr}).\end{aligned} \tag{2.3}$$

It is easy to see that in general both (2.2) and (2.3) also hold for all derived operations t. The proofs go by induction on the complexity of terms. In such a case, we also obtain the inclusion

$$t(A_1,\dots,A_i,\dots,A_n)\cup t(A_1,\dots,B_i,\dots,A_n)\subseteq t(A_1,\dots,A_i\cup B_i,\dots,A_n)$$

that generalizes the distributive law (2.1).

Let $(\mathcal{P}_{>0}M,\Omega,\cup)$ be the extended power algebra of a mode (M,Ω). Recall that such an algebra is very rarely idempotent.

Denote by $\mathcal{I}$ the (quasi)variety of all idempotent τ-algebras of the type $\tau\colon \Omega\uplus\{\cup\}\to\mathbb{N}$. Then $\mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is the set of all congruence relations γ on $(\mathcal{P}_{>0}M,\Omega,\cup)$, such that the quotient $(\mathcal{P}_{>0}M^{\gamma},\Omega)$ is idempotent. By Section 1.4.3 of [5], $\mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is an algebraic subset of the lattice of all congruences of $(\mathcal{P}_{>0}M,\Omega,\cup)$. The least element in $(\mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0}M),\subseteq)$ is called *the $\mathcal{I}$-replica congruence* of $(\mathcal{P}_{>0}M,\Omega,\cup)$. (For more information on replicas, see [21, Section 3.3].)

For $1\le i\le k$ and $k\ge 2$, let t_i be m_i-ary terms. By *the composition term* $t_1\circ t_2\circ\cdots\circ t_k$ of the terms $t_1,t_2,\dots,t_k$ is meant an m-ary term, where $m=m_1\cdots m_k$, defined by the rule:

$$t_1\circ t_2(\overline{x}_1,\dots,\overline{x}_{m_1}) := t_1(t_2(\overline{x}_1),\dots,t_2(\overline{x}_{m_1})),$$
$$t_1\circ\cdots\circ t_k(\overline{x}_1,\dots,\overline{x}_r) := t_1\circ\cdots\circ t_{k-1}(t_k(\overline{x}_1),\dots,t_k(\overline{x}_r)),$$

where $r=m_1\cdots m_{k-1}$ and $\overline{x}_i=(x_{i1},\dots,x_{im_k})$, for $i=1,\dots,r$.

Note that for a mode (M,Ω) and a non-empty subset X of M,

$$t_1\circ\cdots\circ t_k(X,\dots,X)=t_{\sigma(1)}\circ\cdots\circ t_{\sigma(k)}(X,\dots,X),$$

for any permutation σ of the set $\{1,\dots,k\}$. Note also that for any derived operation t, we have $X\subseteq t(X,\dots,X)$. Hence, by (2.2), we can observe the following remark.

Remark 2.1. Let (M,Ω) be a mode. For $1\le i\le k$, let t_i be m_i-ary terms and $\emptyset\ne X\subseteq M$. For the composition term $t=t_1\circ t_2\circ\cdots\circ t_k$, we have

$$t_i(\underbrace{X,X,\dots,X}_{m_i})\subseteq t(\underbrace{X,X,\dots,X}_{m_1\cdots m_k}), \tag{2.4}$$

for each $1\le i\le k$.

Now we define a binary relation ρ on the set $\mathcal{P}_{>0}M$ in the following way:

$$X\ \rho\ Y \iff \begin{array}{l}\text{there are a } k\text{-ary term } t \text{ and an } m\text{-ary term } s,\\ \text{both of type } \Omega, \text{ such that } X\subseteq t(Y,Y,\dots,Y)\\ \text{and } Y\subseteq s(X,X,\dots,X).\end{array} \tag{2.5}$$

Theorem 2.2. *Let (M,Ω) be a mode. The relation ρ is the $\mathcal{I}$-replica congruence of $(\mathcal{P}_{>0}M,\Omega,\cup)$.*

Proof. It is easy to check that ρ is an equivalence relation and that $(\mathcal{P}_{>0}M^{\rho},\Omega)$ is idempotent. We prove that the relation ρ is a congruence on $(\mathcal{P}_{>0}M,\Omega,\cup)$. Let $X,Y,Z,T\in\mathcal{P}_{>0}M$ and $X\ \rho\ Y$ and $Z\ \rho\ W$. This means that there exist k_i-ary terms t_i (for $i=1,2,3,4$), each of type Ω, such that $X\subseteq t_1(Y,Y,\dots,Y)$ and $Y\subseteq t_2(X,X,\dots,X)$, and $Z\subseteq t_3(W,W,\dots,W)$ and $W\subseteq t_4(Z,Z,\dots,Z)$.

Then by Remark 2.1 and (2.3), for $t = t_1 \circ t_3$ we obtain

$$\begin{aligned} X \cup Z &\subseteq t_1(Y, Y, \ldots, Y) \cup t_3(W, W, \ldots, W) \\ &\subseteq t(Y, Y, \ldots, Y) \cup t(W, W, \ldots, W) \subseteq t(Y \cup W, Y \cup W, \ldots, Y \cup W). \end{aligned}$$

Analogously, we can show that $Y \cup W \subseteq s(X \cup Z, \ldots, X \cup Z)$, for $s = t_2 \circ t_4$, so that $(X \cup Z)\ \rho\ (Y \cup W)$.

Now let $\omega \in \Omega$ be an n-ary complex operation and for non-empty subsets X_i, Y_i of M, let $X_i\ \rho\ Y_i$ for each $i = 1, \ldots, n$. We get that for each $i = 1, \ldots, n$, there exist k_i-ary terms t_i and m_i-ary terms s_i, all of them of type Ω, such that $X_i \subseteq t_i(Y_i, \ldots, Y_i)$ and $Y_i \subseteq s_i(X_i, \ldots, X_i)$.

In fact, the idempotent and entropic laws of modes are hyperidentities, so they are satisfied not only by the basic operations, but also by all derived operations. Then by (2.2) and Remark 2.1,

$$\begin{aligned} \omega(X_1, \ldots, X_n) &\subseteq \omega(t_1(Y_1, \ldots, Y_1), \ldots, t_n(Y_n, \ldots, Y_n)) \\ &\subseteq \omega(t(Y_1, \ldots, Y_1), \ldots, t(Y_n, \ldots, Y_n)) \\ &= t(\omega(Y_1, \ldots, Y_n), \ldots, \omega(Y_1, \ldots, Y_n)), \end{aligned}$$

for the composition term $t = t_1 \circ \cdots \circ t_n$.

Similarly, one shows that

$$\omega(Y_1, \ldots, Y_n) \subseteq s(\omega(X_1, \ldots, X_n), \ldots, \omega(X_1, \ldots, X_n)),$$

for $s = s_1 \circ \cdots \circ s_n$. This shows that $\omega(X_1, \ldots, X_n)\ \rho\ \omega(Y_1, \ldots, Y_n)$ and proves that ρ is a congruence relation of $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Further, let $X, Y \in \mathcal{P}_{>0}M$, $X\ \rho\ Y$, and $\gamma \in \mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0}M)$. Then there exist a k-ary term t and an m-ary term s, both of type Ω, such that

$$X \subseteq t(Y, Y, \ldots, Y) \quad \text{and} \quad Y \subseteq s(X, X, \ldots, X).$$

Moreover, for each $\omega \in \Omega$, we have $X\ \gamma\ \omega(X, \ldots, X)$. Hence, $X\ \gamma\ p(X, \ldots, X)$ for any term p of type Ω. This shows that

$$\begin{aligned} X\ \gamma\ s(X, \ldots, X) &= (s(X, \ldots, X) \cup Y)\ \gamma\ (X \cup Y)\ \gamma\ (X \cup t(Y, \ldots, Y)) \\ &= t(Y, \ldots, Y)\ \gamma\ Y, \end{aligned}$$

and finishes the proof. □

Note that if B is a subalgebra of $(\mathcal{P}_{>0}M, \Omega, \cup)$, then the restriction $\rho_B := \rho \cap B^2$ is a congruence on $(B, \Omega, \cup)$. Moreover, for every $X \in B$ and Ω-term t, $t(X, \ldots, X) \in B$. Hence, Theorem 2.2 shows that ρ_B is the $\mathcal{I}$-replica congruence of $(B, \Omega, \cup)$.

Let $\mathcal{P}_{>0}^{<\omega}M$ be the set of all finite non-empty subsets of a mode (M, Ω). Then $\mathcal{P}_{>0}^{<\omega}M$ is a subalgebra of the extended power algebra $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Now we will give another description of the $\mathcal{I}$-replica congruence for the subalgebra $(\mathcal{P}_{>0}^{<\omega}M, \Omega, \cup)$. It will be more convenient to use in the next section.

Let (M,Ω) be a mode, $\emptyset \neq X \subseteq M$, and $\Delta \subseteq \Omega$. For any $n \in \mathbb{N}$, let us define sets $X^{[n]_\Delta}$ in the following way: $X^{[0]_\Delta} := X$, and

$$X^{[n+1]_\Delta} := \bigcup_{\delta \in \Delta} \delta(X^{[n]_\Delta}, \ldots, X^{[n]_\Delta}) = (X^{[n]_\Delta})^{[1]_\Delta}.$$

If $\Delta = \Omega$, we will use the abbreviated notation $X^{[n]}$ instead of $X^{[n]_\Omega}$. It is well known that $\langle X \rangle = \bigcup_{n\in\mathbb{N}} X^{[n]}$, where $\langle X \rangle$ denotes the subalgebra of (M,Ω) generated by X. As proved by A. Romanowska and J. D. H. Smith [19], for each n-ary complex operation $\omega \in \Omega$ and non-empty subsets $X_1, \ldots, X_n$ of M,

$$\langle \omega(X_1, \ldots, X_n) \rangle = \omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle). \tag{2.6}$$

In particular, if $X_1, \ldots, X_n \in \mathcal{P}^{<\omega}_{>0} M$, then the subalgebra $\omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle)$ is finitely generated.

Now we define the second binary relation on the set $\mathcal{P}_{>0} M$:

$$X \;\alpha\; Y \iff \langle X \rangle = \langle Y \rangle. \tag{2.7}$$

Theorem 2.3. *For a mode (M,Ω), the relation α is an element of the set $\mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0} M)$.*

Proof. It is obvious that α is an equivalence relation. To show that it is a congruence relation on the extended power algebra $(\mathcal{P}_{>0} M, \Omega, \cup)$, let $\omega \in \Omega$ be an n-ary complex operation and for non-empty subsets $A_1, \ldots, A_n, B_1, \ldots, B_n \subseteq M$, let $A_i \;\alpha\; B_i$ for $1 \leq i \leq n$. This means that $\langle A_i \rangle = \langle B_i \rangle$ for $1 \leq i \leq n$. Hence,

$$\langle \omega(A_1, \ldots, A_n) \rangle = \omega(\langle A_1 \rangle, \ldots, \langle A_n \rangle) = \omega(\langle B_1 \rangle, \ldots, \langle B_n \rangle) = \langle \omega(B_1, \ldots, B_n) \rangle.$$

Moreover, it is also obvious that $\langle A_1 \cup A_2 \rangle = \langle B_1 \cup B_2 \rangle$. This shows that α is a congruence relation of the algebra $(\mathcal{P}_{>0} M, \Omega, \cup)$. It is easy to observe that $\rho \subseteq \alpha$ and $(\mathcal{P}_{>0} M^{\alpha}, \Omega)$ is idempotent. $\square$

Lemma 2.4. *Let (M,Ω) be a mode, let Δ be a finite subset of Ω, and let $\gamma \in \mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0} M)$. Then $X \;\gamma\; X^{[n]_\Delta}$, for any $n \in \mathbb{N}$.*

Proof. Let $X, Y \in \mathcal{P}_{>0} M$. By assumption, for each $\delta \in \Delta$, we have that $X \;\gamma\; \delta(X, \ldots, X)$. It follows that $X \;\gamma\; \bigcup_{\delta \in \Delta} \delta(X, \ldots, X) = X^{[1]_\Delta}$. Now assume that $X \;\gamma\; X^{[i]_\Delta}$ for some $i > 1$. Then $X \;\gamma\; \bigcup_{\delta \in \Delta} \delta(X^{[i]_\Delta}, \ldots, X^{[i]_\Delta}) = X^{[i+1]_\Delta}$, and consequently $X \;\gamma\; X^{[n]_\Delta}$ for any $n \in \mathbb{N}$. $\square$

Theorem 2.5. *Let (M,Ω) be a mode. The congruences α and ρ restricted to the subalgebra $\mathcal{P}^{<\omega}_{>0} M$ of $(\mathcal{P}_{>0} M, \Omega, \cup)$ coincide: $\alpha_{\mathcal{P}^{<\omega}_{>0} M} = \rho_{\mathcal{P}^{<\omega}_{>0} M}$.*

Proof. Let $X, Y \in \mathcal{P}^{<\omega}_{>0} M$, $X \;\alpha\; Y$, and $\gamma \in \mathrm{Con}_{\mathcal{I}}(\mathcal{P}^{<\omega}_{>0} M)$. It follows that $\bigcup_{n\in\mathbb{N}} X^{[n]} = \langle X \rangle = \langle Y \rangle = \bigcup_{n\in\mathbb{N}} Y^{[n]}$. Since X and Y are finite, there exist $k, l \in \mathbb{N}$ and finite subsets Δ_1, Δ_2 of Ω such that $X \subseteq Y^{[k]_{\Delta_1}}$ and $Y \subseteq X^{[l]_{\Delta_2}}$. By Lemma 2.4, $X \;\gamma\; X^{[l]_{\Delta_2}}$ and $Y \;\gamma\; Y^{[k]_{\Delta_1}}$. Hence,

$$X \;\gamma\; X^{[l]_{\Delta_2}} = X^{[l]_{\Delta_2}} \cup Y \;\gamma\; X \cup Y \;\gamma\; X \cup Y^{[k]_{\Delta_1}} = Y^{[k]_{\Delta_1}} \;\gamma\; Y.$$

So, $\alpha_{\mathcal{P}^{<\omega}_{>0} M} \subseteq \gamma$, and $\alpha_{\mathcal{P}^{<\omega}_{>0} M}$ is the least element of the lattice $\mathrm{Con}_{\mathcal{I}}(\mathcal{P}^{<\omega}_{>0} M)$. $\square$

3. Proof of the main result

As was shown by A. Romanowska and J. D. H. Smith [19], if (M,Ω) is a mode, then the algebra (MS,Ω) of all non-empty subalgebras of (M,Ω) is a mode satisfying each linear identity true in (M,Ω). So, if $\mathcal{M}$ is a variety of modes, then $\mathcal{MS} = \mathrm{HSP}(\{(MS,\Omega) \mid (M,\Omega) \in \mathcal{M}\})$ is also a variety of modes satisfying each linear identity true in $\mathcal{M}$. But the mode (MS,Ω) may also satisfy some non-linear identities which do not follow from linear and idempotent ones.

Example 3.1 ([14])**.** A *differential groupoid* is a mode groupoid $(D,\cdot)$ satisfying the additional linear identity $x(yz) \approx xy$. Note also two other identities true in differential groupoids: $(xy)z \approx (xz)y$ and $x(xy) \approx x$.

Consider the differential groupoid $G = (\{a,b,c,d\},\cdot)$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	d	d	c	c
d	c	c	d	d

The groupoid satisfies the identity: $x \approx (xy)y$ and has 7 subalgebras: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{c,d\}$, G. Note that $(\{a\}G)G = \{a,b\} \neq \{a\}$, so $(GS,\cdot)$ is again a differential groupoid but does not satisfy the identity $x \approx (xy)y$.

However, straightforward calculations show that it satisfies the non-linear identity: $xy \approx ((xy)y)y$, true also in $(G,\cdot)$. But this identity does not follow from any linear and idempotent identity true in $(GS,\cdot)$. Indeed, as it was shown in [16], any linear identity of a differential groupoid may be expressed in the following form

$$(\cdots((x_1x_2)x_3)\cdots)x_n \approx (\cdots((y_1y_2)y_3)\cdots)y_m. \tag{3.1}$$

Assume that an identity (3.1) is true in $(GS,\cdot)$ and let us consider three cases:

Case 1: Let $x_1 \neq y_1$. Then $(GS,\cdot)$ must be trivial, a contradiction.

Case 2: Let $x_1 = y_1$ and $\{x_2,\dots,x_n\} \neq \{y_2,\dots,y_m\}$. Then the identity (3.1) is equivalent to $xy \approx x$. But $(GS,\cdot)$ does not satisfy the latter.

Case 3: Let $x_1 = y_1$ and $\{x_2,\dots,x_n\} = \{y_2,\dots,y_m\}$. It is evident that the identity

$$(\cdots((xx_2)x_3)\cdots)x_n \approx (\cdots((xx_{\pi(2)})x_{\pi(3)})\cdots)x_{\pi(n)} \tag{3.2}$$

holds in any differential groupoid for each permutation π of the set $\{2,\dots,n\}$. But obviously, the identity $xy \approx ((xy)y)y$ could not be obtained from (3.2) by identification of variables.

We will show in this Section that if, for a variety $\mathcal{M}$ of modes, the variety $\mathcal{M}\Sigma$ is locally finite, then the variety $\mathcal{MS}$ satisfies only consequences of linear and idempotent identities true in $\mathcal{M}$.

Definition 3.2. A *modal* is an algebra $(M, \Omega, +)$ such that (M, Ω) is a mode, $(M, +)$ is a (join) semilattice with semilattice order $\leq$, i.e., $x \leq y \Leftrightarrow x + y = y$, and the operations $\omega \in \Omega$ distribute over $+$, i.e.,

$$\omega(x_1, \ldots, x_i + y_i, \ldots, x_n) \approx \omega(x_1, \ldots, x_i, \ldots, x_n) + \omega(x_1, \ldots, y_i, \ldots, x_n).$$

For a given algebra (A, Ω), the set AS of all non-empty subalgebras of (A, Ω) forms a (join) semilattice $(AS, +)$, where $+$ is obtained by setting

$$A_1 + A_2 := \langle A_1 \cup A_2 \rangle, \tag{3.3}$$

for any $A_1, A_2 \in AS$.

A. Romanowska and J. D. H. Smith proved [19] that in the case of modes, these two structures, mode and semilattice, are related by distributive laws. In this way, for all modes (M, Ω), one obtains the algebras $(MS, \Omega, +)$ that provide basic examples of modals. Other examples of modals are given by quotient algebras $(\mathcal{P}_{>0}M^{\gamma}, \Omega, \cup)$, where $\gamma \in \mathrm{Con}_{\mathcal{I}}(\mathcal{P}_{>0}M)$. In what follows, we are interested in the quotient given by the congruence α defined by (2.7).

Theorem 3.3. *Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MS, \Omega, +)$ of all non-empty subalgebras of (M, Ω).*

Proof. Straightforward calculations show that the following mapping:

$$h\colon \mathcal{P}_{>0}M \to MS;\ X \mapsto \langle X \rangle$$

is a surjective homomorphism of the extended power algebra $(\mathcal{P}_{>0}M, \Omega, \cup)$ onto the modal $(MS, \Omega, +)$. Moreover, $\ker h = \alpha$. The result follows by the First Isomorphism Theorem. □

Let MP be the set of all finitely generated subalgebras of a mode (M, Ω). The algebra $(MP, \Omega, +)$ is a subalgebra of the modal $(MS, \Omega, +)$ and for any variety $\mathcal{M}$ of modes, the variety $\mathcal{MP} := \mathrm{HSP}(\{(MP, \Omega) \mid (M, \Omega) \in \mathcal{M}\})$ is a subvariety of $\mathcal{MS}$. In particular, we obtain the following corollary.

Corollary 3.4. *Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}^{<\omega}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MP, \Omega, +)$ of all finitely generated subalgebras of (M, Ω).*

Let $\mathcal{V}$ be a variety of algebras. By Theorem 1.1, the variety

$$\mathcal{V}\Sigma = \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$$

satisfies precisely the consequences of the linear identities holding in $\mathcal{V}$. In fact, the proof of this theorem uses only finite sets. It follows that the same proof applies to its subvariety

$$\mathcal{V}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}_{>0}^{<\omega}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$$

of power algebras of finite subsets.

Corollary 3.5. *Let $\mathcal{V}$ be a variety of algebras. The varieties $\mathcal{V}\Sigma$ and $\mathcal{V}\Sigma_{<\omega}$ coincide.*

Now let $\mathcal{M}$ be a variety of Ω-modes and ρ the $\mathcal{I}$-replica congruence defined as in (2.5). Consider the variety

$$\rho\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M^{\rho}, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\}).$$

In a sequence of lemmas, we show that, with some additional assumptions, this variety is an idempotent subvariety of the variety generated by extended power algebras of finite subsets of modes from $\mathcal{M}$ (see Theorem 3.11).

Lemma 3.6. *Let (M, Ω) be a mode. The $\mathcal{I}$-replica of any homomorphic image of $(\mathcal{P}^{<\omega}_{>0}M, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.*

Proof. By universality property of replication (see [21, Lemma 3.3.1]), each homomorphic image of $(\mathcal{P}^{<\omega}_{>0}M, \Omega, \cup)$ is a homomorphic image of $\mathcal{I}$-replica $(\mathcal{P}^{<\omega}_{>0}M^{\rho}, \Omega, \cup)$. □

Lemma 3.7. *Let (M, Ω) be a mode and B be a subalgebra of $(\mathcal{P}^{<\omega}_{>0}M, \Omega, \cup)$. The $\mathcal{I}$-replica of $(B, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.*

Proof. As we have already noted, for any subalgebra B of $(\mathcal{P}^{<\omega}_{>0}M, \Omega, \cup)$, the restriction $\rho_B := \rho \cap B^2$ is the $\mathcal{I}$-replica congruence of $(B, \Omega, \cup)$. By the Third Isomorphism Theorem (see e.g. [11, Theorem 4.14]), it follows that the quotient $(B^{\rho_B}, \Omega, \cup)$ is isomorphic with a subalgebra of $(\mathcal{P}^{<\omega}_{>0}M^{\rho}, \Omega, \cup)$, which completes the proof. □

Let J be any set, and for each $j \in J$, let $(M_j, \Omega) \in \mathcal{M}$. Directly by Theorem 2.2, one can obtain that the relation $\rho_{\sqcap}$ defined on the set $\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j$ in the following way:

$$\begin{aligned} X \,\rho_{\sqcap}\, Y \iff & \text{there exist a } k\text{-ary } \Omega\text{-term } t \text{ and an } m\text{-ary } \Omega\text{-term } s,\\ & \text{such that, for each } j \in J,\ X(j) \subseteq t(Y(j), \dots, Y(j))\\ & \text{and } Y(j) \subseteq s(X(j), \dots, X(j)) \end{aligned}$$

is the $\mathcal{I}$-replica of $(\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j, \Omega, \cup)$.

Lemma 3.8. *Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set, and for each $j \in J$, let $(M_j, \Omega) \in \mathcal{M}$. The $\mathcal{I}$-replica of $(\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.*

Proof. We will show that the mapping

$$h\colon (\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j)^{\rho_{\sqcap}} \to \prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j^{\rho};\ X^{\rho_{\sqcap}} \mapsto \prod_{j\in J}X(j)^{\rho}$$

is an embedding of $((\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j)^{\rho_{\sqcap}}, \Omega, \cup)$ into $(\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j^{\rho}, \Omega, \cup)$.

First note that the mapping h is well defined. For $X, Y \in \prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j$, let $X^{\rho_{\sqcap}} = Y^{\rho_{\sqcap}}$. This means that there exist a k-ary Ω-term t and an m-ary Ω-term s such that for each $j \in J$:

$$X(j) \subseteq t(Y(j), \dots, Y(j)) \quad \text{and} \quad Y(j) \subseteq s(X(j), \dots, X(j)).$$

Hence, for each $j \in J$, $X(j)^{\rho} = Y(j)^{\rho}$, and finally,

$$h(X^{\rho_{\sqcap}}) = \prod_{j\in J}X(j)^{\rho} = \prod_{j\in J}Y(j)^{\rho} = h(Y^{\rho_{\sqcap}}).$$

Straightforward calculations show that the mapping h is a homomorphism of $((\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j)^{\rho_\sqcap},\Omega,\cup)$ into $(\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j^\rho,\Omega,\cup)$. Now, we have to prove that the mapping h is injective. Let $h(X^{\rho_\sqcap}) = \prod_{j\in J}X(j)^\rho = h(Y^{\rho_\sqcap}) = \prod_{j\in J}Y(j)^\rho$. Thus, for each $j\in J$, there exist a k_j-ary Ω-term t_j and an m_j-ary Ω-term s_j such that:

$$X(j)\subseteq t_j(Y(j),\dots,Y(j)) \quad\text{and}\quad Y(j)\subseteq s_j(X(j),\dots,X(j)).$$

By Remark 2.1, for each $j\in J$, we have

$$X(j)\subseteq t(Y(j),\dots,Y(j)) \quad\text{and}\quad Y(j)\subseteq s(X(j),\dots,X(j))$$

for the composition terms $t = t_1\circ\cdots\circ t_{|J|}$ and $s = s_1\circ\cdots\circ s_{|J|}$. This shows that $X\rho_\sqcap Y$, and $X^{\rho_\sqcap}=Y^{\rho_\sqcap}$, which completes the proof. $\square$

If the set J is not finite, then Lemma 3.8 is no longer true.

Example 3.9. Let $\mathbb{D}$ be the ring of *dyadic* rational numbers, i.e., rationals of the form $m\cdot 2^{-n}$ for integers m and n. Consider the mode $(\mathbb{D},\underline{\mathbb{D}})$, where $\underline{\mathbb{D}}=\{\underline{d}\mid d\in\mathbb{D}\}$ is the set of binary operations defined as follows:

$$\underline{d}(x,y) := (1-d)x+dy.$$

It is known that this algebra belongs to the variety of modes defined by the identities:

$$\underline{0}(x,y)\approx x\approx \underline{1}(y,x) \quad\text{and}\quad \underline{r}(\underline{p}(x,y),\underline{q}(x,y))\approx \underline{r(p,q)}(x,y),$$

and that this variety is equivalent to the variety $\underline{\underline{\mathbb{D}}}$ of affine $\mathbb{D}$-spaces. (See [21, Chapter 6.3].) In particular, each m-ary derived operation $t(x_1,\dots,x_m)$ of $(\mathbb{D},\underline{\mathbb{D}})$ can be expressed as $t(x_1,\dots,x_m) = d_1x_1+\cdots+d_mx_m$, where $d_1,\dots,d_m\in\mathbb{D}$ and $\sum_{i=1}^m d_i = 1$.

Note that for each $n\in\mathbb{N}$, $1=\underline{2^n}(0,\frac{1}{2^n})$ and $\frac{1}{2^n}=\underline{\frac{1}{2^n}}(0,1)$. This implies that $\{0,1\}\subseteq\underline{2^n}(\{0,\frac{1}{2^n}\},\{0,\frac{1}{2^n}\})$ and $\{0,\frac{1}{2^n}\}\subseteq\underline{\frac{1}{2^n}}(\{0,1\},\{0,1\})$. Hence, we obtain that for each $n\in\mathbb{N}$, $\{0,1\}\;\rho\;\{0,\frac{1}{2^n}\}$.

Now for each $n\in\mathbb{N}$, let $M_n:=\mathbb{D}$, and consider $X,Y\in\prod_{n\in\mathbb{N}}\mathcal{P}^{<\omega}_{>0}M_n$, such that $X(n):=\{0,\frac{1}{2^n}\}$ and $Y(n):=\{0,1\}$. Obviously, $\prod_{n\in\mathbb{N}}X(n)^\rho = \prod_{n\in\mathbb{N}}Y(n)^\rho$.

On the other hand, for each m-ary (linear) derived operation t we have

$$\begin{aligned} t(\{0,1\},\dots,\{0,1\}) &= \{t(x_1,\dots,x_m)\mid x_i\in\{0,1\}\}\\ &= \{d_1x_1+\cdots+d_mx_m\mid x_i\in\{0,1\}\}.\end{aligned}$$

Note that $\{d_1x_1+\cdots+d_mx_m\mid x_i\in\{0,1\}\}$ is a finite subset of a commutative submonoid of $(\mathbb{D},+)$ generated by the set $\{d_1,\dots,d_m\}$. It follows that there is no such m-ary term t of type $\underline{\mathbb{D}}$ such that $\bigcup_{n\in\mathbb{N}}\{0,\frac{1}{2^n}\}\subseteq t(\{0,1\},\{0,1\},\dots,\{0,1\})$. This shows that $X^{\rho_\sqcap}\neq Y^{\rho_\sqcap}$, and in consequence, the mapping

$$h\colon (\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j)^{\rho_\sqcap}\to\prod_{j\in J}\mathcal{P}^{<\omega}_{>0}M_j^\rho;\; X^{\rho_\sqcap}\mapsto\prod_{j\in J}X(j)^\rho$$

is not injective.

Let $\mathcal{M}\Sigma^{\cup}_{<\omega}$ denote the variety generated by extended power algebras of finite subsets of algebras from $\mathcal{M}$, i.e.,

$$\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(P^{<\omega}_{>0}M, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\}),$$

and let $\mathcal{IM}\Sigma^{\cup}_{<\omega}$ denote the idempotent subvariety of $\mathcal{M}\Sigma^{\cup}_{<\omega}$. It is well known that

$$\mathcal{IM}\Sigma^{\cup}_{<\omega} = \mathrm{HSP}(\{F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n) \mid n \in \mathbb{N}\}),$$

where $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ denotes the free $\mathcal{IM}\Sigma^{\cup}_{<\omega}$-algebra on n generators. Obviously, each free algebra $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ is the idempotent replica of the free $\mathcal{M}\Sigma^{\cup}_{<\omega}$-algebra $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$.

Lemma 3.10. *Let the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ be locally finite. The idempotent replica of $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.*

Proof. If the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite, then for each $n \in \mathbb{N}$, $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n) \in \mathrm{HSP}_{fin}(\{(P^{<\omega}_{>0}M, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\})$. Hence, by Lemmas 3.6–3.8, the idempotent replica of $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$. □

This implies the following theorem.

Theorem 3.11. *Let $\mathcal{M}$ be a variety of Ω-modes such that $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. Then*

$$\mathcal{IM}\Sigma^{\cup}_{<\omega} = \rho\mathcal{M}\Sigma^{\cup}_{<\omega}.$$

By Corollary 3.4 and Theorem 2.5, the following three varieties of modals: $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$, $\mathrm{HSP}(\{(MP, \Omega, +) \mid (M, \Omega) \in \mathcal{M}\})$, and $\mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M^{\alpha}, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\})$, coincide.

In particular, they satisfy the same identities that do not involve the additional operation of the union. Hence, for varieties

$$\rho\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M^{\rho}, \Omega) \mid (M, \Omega) \in \mathcal{M}\}),$$
$$\alpha\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M^{\alpha}, \Omega) \mid (M, \Omega) \in \mathcal{M}\}),$$

we obtain

$$\mathcal{M}\Sigma = \mathcal{M}\Sigma_{<\omega} \supseteq \rho\mathcal{M}\Sigma_{<\omega} = \alpha\mathcal{M}\Sigma_{<\omega} = \mathcal{MP}.$$

Theorem 3.12. *Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. Then $\mathcal{MS} = \mathcal{MP}$.*

Proof. Theorem 3.11 shows that in the case when $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite, the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$ is defined precisely by idempotent laws and the basis of $\mathcal{M}\Sigma^{\cup}_{<\omega}$. By Theorem 1.1 and Corollary 3.5, the basis of $\mathcal{M}\Sigma^{\cup}_{<\omega}$ includes only linear identities that do not involve the additional operation $\cup$. Hence, the variety $\rho\mathcal{M}\Sigma_{<\omega} = \mathcal{MP}$ is defined precisely by the idempotent and linear identities true in $\mathcal{M}$.

On the other hand, the variety $\mathcal{MS}$ preserves all idempotent and linear identities of $\mathcal{M}$ (see [19], [18, Section 5.1] or [12]), so $\mathcal{MS} \subseteq \mathcal{MP}$. But because $\mathcal{MS}$ includes $\mathcal{MP}$ as a subvariety, it follows that $\mathcal{MS} = \mathcal{MP}$. □

Finally, we will show that for any variety $\mathcal{M}$ of modes, the variety $\mathcal{M}\Sigma$ is locally finite if and only if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite.

Let $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a set $X \subseteq A$. An element $a \in A$ is said to be in *disjunctive form* if it is a union of a finite number of elements from $\langle X \rangle$, where $\langle X \rangle$ denotes the subalgebra of (A, Ω) generated by X. The following lemma shows that each element in $(A, \Omega, \cup)$ may be expressed in such form.

Lemma 3.13. (Disjunctive Form Lemma). *Let $\mathcal{M}$ be a variety of Ω-modes and $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a set $X \subseteq A$. For each $a \in A$, there exist $a_1, \dots, a_p \in \langle X \rangle$ such that $a = a_1 \cup \cdots \cup a_p$.*

Proof. The proof goes by induction on the minimal number m of occurrences of the semilattice operation $\cup$ in the expression of a as a $\mathcal{M}\Sigma^{\cup}_{<\omega}$-word in the alphabet X.

Consider $a = a_1$ with $a_1 \in \langle X \rangle$. Hence, the result holds for $m = 0$. Now suppose that the hypothesis is established for $m > 0$, and let $a \in A$ be an element in which the semilattice operation $\cup$ occurs $m + 1$ times. If $a = a_1 \cup a_2$ for some $a_1, a_2 \in A$, then by the induction hypothesis, there are $a_{11}, \dots, a_{1k}, a_{21}, \dots, a_{2n} \in \langle X \rangle$ such that

$$a = a_1 \cup a_2 = a_{11} \cup \cdots \cup a_{1k} \cup a_{21} \cup \cdots \cup a_{2n}.$$

Otherwise, $a = \omega(a_1, \dots, a_k \cup b_k, \dots, a_n)$ for some $\omega \in \Omega$ and $a_1, \dots, a_k, \dots, a_n$, $b_k \in A$. Then by distributivity, we have

$$a = \omega(a_1, \dots, a_k \cup b_k, \dots, a_n) = \omega(a_1, \dots, a_k, \dots, a_n) \cup \omega(a_1, \dots, b_k, \dots, a_n).$$

Because words $\omega(a_1, \dots, a_k, \dots, a_n)$ and $\omega(a_1, \dots, b_k, \dots, a_n)$ are in A and each of them has at most m occurrences of the operation $\cup$, this completes the inductive proof. □

Theorem 3.14. *Let $\mathcal{M}$ be a variety of Ω-modes. The variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite if and only if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite.*

Proof. Let $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a finite set $X \subseteq A$. By the Disjunctive Form Lemma 3.13, for each $a \in A$, there exist $a_1, \dots, a_p \in \langle X \rangle$ such that

$$a = a_1 \cup \cdots \cup a_p. \tag{3.4}$$

If the variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite, then the algebra $\langle X \rangle \in \mathcal{M}\Sigma_{<\omega}$ is finite. Hence, there are only finitely many elements of the form (3.4). Consequently, the algebra $(A, \Omega, \cup)$ is finite.

Let $(F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(X), \Omega, \cup)$ be the free algebra in the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$. It is known that free algebra over X in the variety generated by Ω-subreducts of algebras in $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is isomorphic to the Ω-subreduct $(\langle X \rangle, \Omega)$, generated by X, of the free algebra $(F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(X), \Omega, \cup)$. (See e.g. [14, Theorem 3.9]). The free algebra $(F_{\mathcal{M}\Sigma_{<\omega}}(X), \Omega)$ is then a homomorphic image of $(\langle X \rangle, \Omega)$. Consequently, the variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. □

Theorem 3.12, Corollary 3.5 and Theorem 3.14 imply our main result.

Main Theorem. *Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite. The variety $\mathcal{M}\mathcal{S}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.*

Example 3.15. Let $\mathcal{M}\Sigma$ be the entropic variety of groupoids $(G, \cdot)$ satisfying the following linear identities:

$$(xy \cdot z) \cdot t \approx xy \cdot z \quad \text{and} \quad t \cdot (z \cdot xy) \approx tz \cdot xy.$$

Note that the identity $xy \cdot z \approx xa \cdot b$ is also true in $\mathcal{M}\Sigma$. It implies

$$x \cdot (y \cdot zt) \approx (x \cdot yz) \cdot t \approx x \cdot (yz \cdot t) \approx (xy \cdot z) \cdot t \approx xx \cdot x.$$

This shows that the variety $\mathcal{M}\Sigma$ is locally finite and the idempotent replica of the free $\mathcal{M}\Sigma$-groupoid is different from the left-zero or the right-zero semigroup. The Main Theorem shows that for any idempotent subvariety $\mathcal{M} \subseteq \mathcal{M}\Sigma$, $\mathcal{M}\mathcal{S}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.

As an immediate consequence of the Main Theorem, we obtain the confirmation of Conjecture 1.3 in the case of a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite.

Corollary 3.16. *Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite. Then $\mathcal{M} = \mathcal{M}\mathcal{S}$ if and only if $\mathcal{M}$ is defined only by idempotent and linear identities.*

Our Main Theorem solved the problem in the case the variety $\mathcal{M}\Sigma$ is locally finite. But there remains the question whether the result is true in the more general case. Though Example 3.9 may suggest a negative answer, we still do not know if Conjecture 1.3 is true for an arbitrary variety $\mathcal{M}$ of modes. On the other hand, Example 3.17 below shows that the assumption of the local finiteness of $\mathcal{M}\Sigma$ is not essential.

Example 3.17. Each proper non-trivial subvariety of the variety $\mathcal{D}$ of differential groupoids (defined as in Example 3.1) is relatively based by a unique identity of the form

$$(\cdots((x\underbrace{y)y)\cdots)y}_{i \text{ times}} =: xy^i \approx xy^{i+j}$$

for some $i \in \mathbb{N}$ and positive integer j (see [16]). Denote such a variety by $\mathcal{D}_{i,i+j}$. It was proved in [12] that $\mathcal{D}_{i,i+j}\mathcal{S} = \mathcal{D}$ for each i and j, but the variety

$\mathcal{D}\Sigma$ is not locally finite since it includes as a subvariety the non-locally finite variety $\mathcal{D}$.

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